

Inverse Diffusivity Problem via Homogenization Theory

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Abstract

Polarization tensor corresponding to near zero volume inhomogeneities was introduced in the pioneering work by Capdeboscq-Vogelius [9, 10]. A beautiful application of the polarization tensor to an inverse problem involving inhomogeneities was also given by them. In this article, we take an approach toward polarization tensor via homogenized tensor. Accordingly, we introduce polarization tensor corresponding to inhomogeneities with positive volume fraction. A relation between this tensor and the homogenized tensor is found. Next, we proceed to examine the sense in which this tensor is continuous as the volume fraction tends to zero. Our approach has its own advantages, as we will see. In particular, it provides another method to deduce optimal estimates on polarization tensors in any dimension from those on homogenized tensors, along with the information on underlying microstructures.

Keywords: Polarization tensors, Homogenization theory, Calderón problem

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1 Introduction and statement of main results

We consider a conducting object that occupies a bounded open set $\Omega \subset \mathbb{R}^N$ with smooth boundary. Let γ_0 denotes the constant background conductivity of the object in the absence of any inhomogeneities. Let ω_ϵ denote a set of inhomogeneities inside Ω , we assume that the set ω_ϵ is measurable for each ϵ and separated away from the boundary, that is, $\text{dist}(\omega_\epsilon, \partial\Omega) \geq d_0 > 0$. We assume that $|\omega_\epsilon| \rightarrow \delta \geq 0$ as $\epsilon \rightarrow 0$. Such a situation was considered by Capdeboscq-Vogelius in [9, 10] with $\delta = 0$.

Let γ_ϵ denote the two-phase (γ_0, γ_1) conductivity profile of the medium in the presence of inhomogeneities, that is

$$\gamma_\epsilon(x) = \gamma_1 \chi_{\omega_\epsilon}(x) + \gamma_0(1 - \chi_{\omega_\epsilon}(x)) \quad x \in \Omega. \quad (1)$$

We assume that $0 < \gamma_1 < \gamma_0 < \infty$. (The case $\gamma_0 < \gamma_1$ can be treated analogously).

The voltage potential in the above two phase medium is denoted by $u_\epsilon(x)$. It is the solution to the following two phase homogenization problem:

$$\begin{aligned}\nabla \cdot (\gamma_\epsilon(x) \nabla u_\epsilon(x)) &= 0 \text{ in } \Omega \\ u_\epsilon(x) &= f(x) \text{ on } \partial\Omega \quad \text{for } f \in H^{1/2}(\partial\Omega).\end{aligned}\tag{2}$$

Let $M(\gamma_1, \gamma_0; \Omega)$ denote the set of all real symmetric positive definite matrices lying between γ_1 and γ_0 . Assume that there exist $\theta \in L^\infty(\Omega; [0, 1])$ and $\gamma^* \in M(\gamma_1, \gamma_0; \Omega)$, such that

$$\chi_{\omega_\epsilon}(x) \rightharpoonup \theta(x) \quad \text{weak}^* \text{ in } L^\infty(\Omega; [0, 1])\tag{3}$$

and

$$\gamma_\epsilon(x)I \text{ H-converge to } \gamma^*(x) \in M(\gamma_1, \gamma_0; \Omega)\tag{4}$$

in the sense that,

$$\begin{aligned}u_\epsilon &\rightharpoonup u \quad \text{weakly in } H^1(\Omega), \\ \gamma_\epsilon \nabla u_\epsilon &\rightharpoonup \gamma^* \nabla u \quad \text{weakly in } (L^2(\Omega))^N\end{aligned}\tag{5}$$

where u is the solution of the homogenized equation :

$$\begin{aligned}\nabla \cdot (\gamma^* \nabla u(x)) &= 0 \quad \text{in } \Omega \\ u(x) &= f(x) \quad \text{on } \partial\Omega.\end{aligned}\tag{6}$$

It is known above convergences (3),(4) hold for a suitable subsequence. Our hypothesis here is that they hold for the entire sequence. Motivated by the inverse problem of determining the measure of dilute inhomogeneities, the authors of [9, 10] consider the equation in (2) with Neumann boundary condition and the asymptotic observation/measurement which is nothing but the difference in potential on the boundary. Since we have imposed the Dirichlet boundary condition in the problem (2), we consider the following asymptotic observation/measurement which is nothing but the current perturbation on the boundary of the domain :

$$(\gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} - \gamma_0 \frac{\partial u}{\partial \nu})|_{\partial\Omega} \text{ as } \epsilon \rightarrow 0,\tag{7}$$

where ν is the outer unit normal to $\partial\Omega$. A general asymptotic formula for boundary current perturbations under the condition that the volume of inhomogeneities goes to zero was derived in [9]. For earlier works in this direction under certain restrictions on the inhomogeneities, we refer [17, 8, 5, 7]. The asymptotic formula derived in [9, 10] involved the so-called polarization tensors which form a set of macro coefficients associated to dilute inhomogeneities. They studied several properties of these tensors in [9, 10, 11, 12, 13]. For a comprehensive treatment of polarization tensors, we refer the reader to [4].

In this article, as already mentioned, we consider inhomogeneities with total volume $\delta > 0$. The appropriate quantity for which we seek the asymptotic formula is then the following :

$$(\gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} - \gamma^* \frac{\partial u}{\partial \nu})|_{\partial\Omega} \text{ as } \epsilon \rightarrow 0,\tag{8}$$

This generalizes (7) because when $\delta = 0$ the homogenized tensor γ^* coincides with $\gamma_0 I$. Naturally the above formula involves a new polarization tensor corresponding to δ positive. Roughly speaking, polarization tensor provides first order approximation of the homogenized tensor, γ^* for small volume proportion of inhomogeneities [4, 9, 14, 15]. The papers [9, 10] present a microscopic interpretation of the polarization tensor. To establish its regularity properties near zero-volume fraction, we find it appropriate to introduce the polarization tensor at non-zero volume fraction and then study its asymptotic properties as volume fraction goes to zero. Our approach examines the precise sense in which polarization tensor is continuous at zero volume fraction. It has also other advantages as well. One advantage is that we obtain a relation between polarization tensor and homogenized tensor for non-zero volume fraction and another advantage is that it allows to use the knowledge about the homogenized tensor to deduce the properties of polarization tensor. As an example, we deduce optimal estimates on polarization tensors in any dimension from those on homogenized tensors, along with the underlying microstructures. At the core of our approach lies the following approximation result : any polarization tensor corresponding to zero volume fraction can be obtained as an appropriate limit of polarization tensor corresponding to non-zero volume fraction (Theorem 1.2).

We close this section by stating how the article is organized. In the reminder of this section we state our main results. The first theorem gives asymptotic expression for the perturbed current (8) as $|\omega_\epsilon| \rightarrow \delta$. The proof of this Theorem is presented in Section 2. This naturally calls the introduction of the polarization tensor denoted M^θ of the non-zero volume fraction. An important relation linking this polarization tensor with homogenized tensor γ^* is stated in Proposition 2.2. Bounds on M^θ are easily deduced are those of γ^* . Theorem 1.2 is concerned with a continuity property of polarization tensor whose proof is presented in Section 3. Theorem 1.3 presents the optimal bounds in terms of trace inequalities on polarization tensor near zero volume fraction as a consequence of Theorem 1.2. Regarding the microstructures underlying polarization tensors M^0 with near zero volume inhomogeneities, we recall that [11] shows that equality in the above trace bounds holds for the so-called “washers” microstructures in two dimension. Numerical evidence for the same is provided in [3]. Examples of “thin” inhomogeneities are treated in [12]. Thanks to our approach, we are able to compute M^0 corresponding to sequential laminates of any rank in any dimension with relative ease. Such tensors “fill up” the region in the phase space defined by the trace inequalities (14). See Section 4. Such a computation seems hard without passing through the homogenized tensor.

In order to state our result, we require a few preliminaries from homogenization theory [1, 16]. The homogenized tensor is obtained from oscillating test functions which are defined by

$$\begin{aligned} -\nabla \cdot (\gamma_\epsilon(x) \nabla w_\epsilon^i(x)) &= -\operatorname{div}(\gamma^* e_i) \quad \text{in } \Omega, \\ w_\epsilon^i &= x_i \quad \text{on } \partial\Omega. \end{aligned} \tag{9}$$

The matrix W_ϵ defined by its columns $(\nabla w_\epsilon^i)_{1 \leq i \leq N}$ is called the corrector matrix with the

following properties:

$$\begin{aligned} W_\epsilon &\rightharpoonup I \text{ weakly in } L^2(\Omega)^{N \times N} \\ \gamma_\epsilon(x)W_\epsilon(x) &\rightharpoonup \gamma^* \text{ weakly in } L^2(\Omega)^{N \times N}. \end{aligned} \quad (10)$$

We can write

$$\nabla u_\epsilon = W_\epsilon \nabla u + r_\epsilon \quad (11)$$

where $r_\epsilon \rightarrow 0$ strongly in $(L^1_{\text{loc}}(\Omega))^N$.

We also need the so-called boundary Green's function which is defined as follows. For $y \in \partial\Omega$, consider

$$\begin{aligned} \nabla \cdot (\gamma_0 \nabla_x D(x, y)) &= 0 \text{ in } \Omega \\ D(x, y) &= \delta_y(x) \text{ on } \partial\Omega. \end{aligned} \quad (12)$$

THEOREM 1.1. *Let $\delta > 0$. Given $f \in H^{1/2}(\partial\Omega)$, let u and u_ϵ denote the solutions to (6) and (2) respectively. Then there exists a subsequence (still denoted by ϵ), a regular positive compactly supported Radon measure μ^θ , and a matrix-valued function $M^\theta \in L^2(\Omega, d\mu^\theta)$ (called polarization tensor) such that*

$$\begin{aligned} (\gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} - \gamma^* \frac{\partial u}{\partial \nu})(y) &= |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M_{ij}^\theta(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial D}{\partial x_j}(x, y) d\mu^\theta(x) \\ &\quad + \int_\Omega (\gamma_0 I - \gamma^*(x))_{ij} \frac{\partial u}{\partial x_i}(x) \frac{\partial D}{\partial x_j}(x, y) dx + o(1). \end{aligned} \quad (13)$$

The $o(1)$ term goes to zero uniformly in y as ϵ goes to zero.

REMARK 1.1. *The polarization tensor M^θ depends on the microstructures ω_ϵ under consideration. Considering γ_ϵ in the case of $\delta = 0$ the above Theorem 1.1 was proved in [9].*

REMARK 1.2. *If we consider periodic microstructures with a given volume fraction of inhomogeneities, then $\theta > 0$ is constant which is equal to the volume fraction and the corresponding polarization tensor M^θ is also constant. Moreover, the corresponding measure μ^θ is given by $\frac{1}{|\Omega|} dx$ (see (5)).*

THEOREM 1.2. *For a given $d\mu^0$, M^0 , the polarization tensor of near zero-volume fraction with $M^0 \in L^2(\Omega, d\mu^0)$, for any point $x_0 \in \text{support of } d\mu^0$, μ^0 almost everywhere there exists a sequence $\theta_{x_0}^n \in (0, 1]$ depending upon the point x_0 and a sequence of polarization tensors $M^{\theta_{x_0}^n}$ which are constant, such that as $n \rightarrow \infty$, $\theta_{x_0}^n \rightarrow 0$ and $M^{\theta_{x_0}^n} \rightarrow M^0(x_0)$, μ^0 almost everywhere x_0 .*

THEOREM 1.3. (a): *Let M^0 denote a polarization tensor corresponding to zero volume fraction. Then for μ^0 almost everywhere $x \in \text{support of } \mu^0$, we have the following pointwise trace bounds,*

$$\begin{aligned} \text{Lower Bound : } \quad \text{trace}(M^0(x))^{-1} &\leq (N-1) + \frac{\gamma_1}{\gamma_0} \\ \text{Upper Bound : } \quad \text{trace}(M^0(x)) &\leq (N-1) + \frac{\gamma_0}{\gamma_1}. \end{aligned} \quad (14)$$

(b): These bounds are optimal in the sense that any $(\lambda_1(x), \dots, \lambda_N(x))$ satisfying pointwise

$$\begin{aligned} \sum_{i=1}^N \lambda_i(x) &\leq (N-1) + \frac{\gamma_0}{\gamma_1}. \\ \sum_{i=1}^N \frac{1}{\lambda_i(x)} &\leq (N-1) + \frac{\gamma_1}{\gamma_0}. \end{aligned}$$

arises as the eigenvalues of a polarization tensor of zero volume fraction at that point x .

2 Proof of Theorem 1.1

In this section, we obtain an asymptotic formula for the boundary current perturbations when the volume of inhomogeneities goes to $\delta > 0$ as $\epsilon \rightarrow 0$.

We begin by applying the divergence formula a few times to obtain an expression for the boundary current perturbation.

Multiply $D(x, y)$ in (2) and (6) and using the divergence formula, we have

$$\begin{aligned} \int_{\Omega} \gamma_{\epsilon}(x) \nabla u_{\epsilon}(x) \cdot \nabla_x D(x, y) dx &= \int_{\partial\Omega} \gamma_{\epsilon}(x) \frac{\partial u_{\epsilon}}{\partial \nu}(x) D(x, y) d\sigma(x) \\ \int_{\Omega} \gamma^*(x) \nabla u(x) \cdot \nabla_x D(x, y) dx &= \int_{\partial\Omega} \gamma^*(x) \frac{\partial u}{\partial \nu}(x) D(x, y) d\sigma(x) \end{aligned}$$

Subtracting these two equations, we get

$$\begin{aligned} ((\gamma_{\epsilon}(x) \frac{\partial u_{\epsilon}}{\partial \nu}(x) - \gamma^*(x) \frac{\partial u}{\partial \nu}(x))) D(x, y)|_{\partial\Omega} &= \gamma_0 (\frac{\partial u_{\epsilon}}{\partial \nu} - \frac{\partial u}{\partial \nu})(y) \\ &= \int_{\Omega} \gamma_{\epsilon}(x) \nabla u_{\epsilon}(x) \cdot \nabla_x D(x, y) dx - \int_{\Omega} \gamma^*(x) \nabla u(x) \cdot \nabla_x D(x, y) dx \\ &= \int_{\omega_{\epsilon}} (\gamma_1 - \gamma_0) \nabla u_{\epsilon}(x) \cdot \nabla_x D(x, y) dx + \int_{\Omega} [\gamma_0 \nabla u_{\epsilon} - \gamma^*(x) \nabla u(x)] \cdot \nabla_x D(x, y) dx. \end{aligned} \tag{1}$$

Now multiplying (12) by u_{ϵ} and u and again applying the divergence formula, we have,

$$\begin{aligned} \int_{\Omega} \gamma_0 \nabla_x D(x, y) \cdot \nabla u_{\epsilon}(x) dx &= \int_{\partial\Omega} \gamma_0 \frac{\partial D(x, y)}{\partial \nu}(x) u_{\epsilon}(x) d\sigma(x) \\ \int_{\Omega} \gamma_0 \nabla_x D(x, y) \cdot \nabla u(x) dx &= \int_{\partial\Omega} \gamma_0 \frac{\partial D(x, y)}{\partial \nu}(x) u(x) d\sigma(x). \end{aligned}$$

Since $u_{\epsilon} = u = f$ on $\partial\Omega$, we have

$$\int_{\Omega} \gamma_0 \nabla_x D(x, y) \cdot \nabla u_{\epsilon}(x) dx = \int_{\Omega} \gamma_0 \nabla_x D(x, y) \cdot \nabla u(x) dx.$$

Therefore

$$\begin{aligned} (\gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} - \gamma^* \frac{\partial u}{\partial \nu})(y) &= \gamma_0 (\frac{\partial u_\epsilon}{\partial \nu} - \frac{\partial u}{\partial \nu})(y) = \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \nabla u_\epsilon(x) \cdot \nabla_x D(x, y) dx \\ &\quad + \int_{\Omega} (\gamma_0 I - \gamma^*(x)) \nabla u(x) \cdot \nabla_x D(x, y) dx. \end{aligned} \quad (2)$$

By (11), we can rewrite

$$\begin{aligned} \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \nabla u_\epsilon(x) \cdot \nabla_x D(x, y) dx &= \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) (W_\epsilon \nabla u + r_\epsilon) \cdot \nabla_x D(x, y) dx \\ &= |\omega_\epsilon| \int_{\Omega} (\gamma_1 - \gamma_0) \frac{\chi_{\omega_\epsilon}(x)}{|\omega_\epsilon|} \frac{\partial w_\epsilon^i}{\partial x_j} \frac{\partial u}{\partial x_j} \frac{\partial D(x, y)}{\partial x_i} dx \\ &\quad + \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) r_\epsilon \cdot \nabla_x D(x, y) dx. \end{aligned} \quad (3)$$

We first focus on (3) above. We have the following results.

PROPOSITION 2.1. *Let $\{w_\epsilon\}$ be as in (9). Then*

$$\|\nabla w_\epsilon^i - e_i\|_2 \leq C |\omega_\epsilon|^{\frac{1}{2}} + \|(\gamma^* - \gamma_0 I) e_i\|_2. \quad (4)$$

Proof. Consider the corrector equation (9) and multiply by $(w_\epsilon^i - x_i)$ and using divergence formula, we have

$$\int_{\Omega} \gamma_\epsilon \nabla w_\epsilon^i \cdot (\nabla w_\epsilon^i - e_i) dx = \int_{\Omega} \gamma^*(x) (\nabla w_\epsilon^i - e_i) dx.$$

Rewriting it as,

$$\begin{aligned} \int_{\Omega} \gamma_\epsilon (\nabla w_\epsilon^i - e_i) \cdot (\nabla w_\epsilon^i - e_i) dx &= \int_{\Omega} (\gamma^* - \gamma_0 I) e_i \cdot (\nabla w_\epsilon^i - e_i) dx \\ &\quad - \int_{\Omega} (\gamma_1 - \gamma_0) \chi_{\omega_\epsilon} e_i \cdot (\nabla w_\epsilon^i - e_i) dx. \end{aligned}$$

■

REMARK 2.1. *If $\delta = 0$ then as $\gamma^* = \gamma_0 I$, in that case (4) becomes $\|\nabla w_\epsilon^i - e_i\|_2 \leq C |\omega_\epsilon|^{\frac{1}{2}}$.*

LEMMA 2.1. *Let $\{w_\epsilon\}$ be as in (9). There exists a subsequence still denoted by ϵ , function $\theta \in L^\infty(\Omega, [0, 1])$, compactly supported a positive Radon measure μ^θ , and a 2-tensor $M \in L^2(\Omega, d\mu^\theta)$ such that*

$$\begin{aligned} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) dx &\xrightarrow{*} d\mu^\theta \text{ in } (C^0(\overline{\Omega}))^*, \\ \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \frac{\partial w_\epsilon^i}{\partial x_j} dx &\xrightarrow{*} M_{ij}^\theta d\mu^\theta \text{ in } (C^0(\overline{\Omega}))^*. \end{aligned}$$

Proof. Since $\chi_{\omega_\epsilon}(x) \in L^\infty(\Omega, \{0, 1\})$ there exists a $\theta \in L^\infty(\Omega, [0, 1])$ and a subsequence such that

$$\chi_{\omega_\epsilon}(x) \xrightarrow{*} \theta(x) \text{ in } L^\infty(\Omega, [0, 1]).$$

In particular, we have

$$|\omega_\epsilon| = \int_{\Omega} \chi_{\omega_\epsilon}(x) dx \rightarrow \int_{\Omega} \theta(x) dx = \delta.$$

Now since $\frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x)$ is bounded in $L^1(\Omega)$, from Banach-Alaoglu theorem and Riesz Representation theorem, there exists a regular, Radon measure μ^θ and a subsequence (denoted by ϵ) such that

$$\frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \xrightarrow{*} d\mu^\theta \text{ in } (C^0(\bar{\Omega}))^*.$$

And if $\delta > 0$ we see

$$d\mu^\theta = \frac{\theta(x)}{\int_{\Omega} \theta(x) dx} dx \text{ almost everywhere } x \text{ in } \Omega. \quad (5)$$

Now, $\frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \nabla w_\epsilon^i$ is also bounded $L^1(\Omega)$ for each $i = 1, \dots, N$ as

$$\begin{aligned} \left\| \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \nabla w_\epsilon^i \right\|_{L^1(\Omega)} &= \int_{\Omega} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) |(\nabla w_\epsilon^i - e_i) + e_i| dx \\ &\leq \frac{1}{|\omega_\epsilon|^{\frac{1}{2}}} \left(\int_{\Omega} |\nabla w_\epsilon^i - e_i|^2 dx \right)^{1/2} + \frac{1}{|\omega_\epsilon|^{\frac{1}{2}}} \left(\int_{\omega_\epsilon} 1 dx \right)^{1/2} \\ &\leq C \text{ by using (4) for both } \delta = 0 \text{ and } \delta > 0 \text{ cases.} \end{aligned}$$

So there exists a subsequence and a Radon measure dM_{ij}^θ such that

$$\frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \frac{\partial w_\epsilon^i}{\partial x_j} dx \xrightarrow{*} dM_{ij}^\theta \text{ in } (C^0(\bar{\Omega}))^*. \quad (6)$$

Now,

$$\begin{aligned} \left| \int_{\Omega} \phi(x) dM_{ij}^\theta \right| &= \left| \lim \int_{\Omega} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \frac{\partial w_\epsilon^i}{\partial x_j} \phi(x) dx \right| \\ &\leq \varliminf \left(\frac{1}{|\omega_\epsilon|} \right)^{1/2} \left(\int_{\Omega} \left| \frac{\partial w_\epsilon^i}{\partial x_j} \right|^2 dx \right)^{1/2} \left(\int_{\Omega} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) |\phi|^2 dx \right)^{1/2} \\ &\leq C \left(\int_{\Omega} \theta(x) dx \right)^{-1/2} \left(\int_{\Omega} |\phi|^2 d\mu^\theta \right)^{1/2} \text{ for all } \phi \in C^0(\bar{\Omega}). \end{aligned}$$

Therefore, $\phi \mapsto \int_{\Omega} \phi dM_{ij}^\theta$ can be extended to a bounded linear functional on $L^2(\Omega, d\mu^\theta)$. Now by Riesz-Representation theorem

$$\int_{\Omega} \phi dM_{ij}^\theta = \int_{\Omega} \phi M_{ij}^\theta d\mu^\theta \text{ for some function } M_{ij}^\theta \in L^2(\Omega, d\mu^\theta).$$

Hence,

$$dM_{ij}^\theta = M_{ij}^\theta d\mu^\theta.$$

For $\delta > 0$ we also see that

$$dM_{ij}^\theta = M_{ij}^\theta d\mu^\theta = \frac{M_{ij}^\theta(x)\theta(x)}{\int \theta(x)dx} dx \quad \text{almost everywhere in } \Omega. \quad (7)$$

This completes the proof of the lemma. ■

REMARK 2.2. We call the matrix $M^\theta = (M_{ij}^\theta)$ polarization tensor with non-zero volume fraction. For $\delta = 0$, we have $\gamma^* = \gamma_0 I$, and the measure μ^0 and the matrix M^0 are given by (cf. [9]).

$$\frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \frac{\partial w_\epsilon^i}{\partial x_j} dx \xrightarrow{*} dM_{ij}^0 = M_{ij}^0 d\mu^0 \text{ in } (C^0(\overline{\Omega}))^* \quad (8)$$

with

$$\frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \xrightarrow{*} d\mu^0 \text{ in } (C^0(\overline{\Omega}))^* \text{ and } M^0 \in L^2(\Omega, d\mu^0). \quad (9)$$

The correctors $(w_\epsilon^i(x))_{1 \leq i \leq N}$ are the solution of

$$-\nabla \cdot (\gamma_\epsilon(x) \nabla w_\epsilon^i(x)) = -\text{div}(\gamma_0 e_i) \text{ in } \Omega, \quad w_\epsilon^i(x) = x_i \text{ on } \partial\Omega. \quad (10)$$

REMARK 2.3. For $\delta > 0$, (6) shows that product of two weakly convergent sequence $\frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon} dx$ and ∇w_ϵ^i converges weakly and the limit is not the product of limits. The measure μ^θ is absolutely continuous with respect to Lebesgue measure. M^θ is defined almost everywhere in Ω with respect to the Lebesgue measure. Such properties do not hold for μ^0, M^0 . For $\delta = 0$, M^0 is defined over set of the support of $d\mu^0$. As a convention we define M^0 equal to identity elsewhere in Ω .

We now prove Theorem 1.1.

Proof of Theorem 1.1. Let $y \in \partial\Omega$. We recall the following two equations:

$$\begin{aligned} \gamma_0 \left(\frac{\partial u_\epsilon}{\partial \nu} - \frac{\partial u}{\partial \nu} \right)(y) &= \underbrace{\int_{\omega_\epsilon} (\gamma_1 - \gamma_0) \nabla u_\epsilon(x) \cdot \nabla_x D(x, y) dx}_J \\ &\quad + \int_{\Omega} (\gamma_0 I - \gamma^*(x)) \nabla u(x) \cdot \nabla_x D(x, y) dx \end{aligned} \quad (11)$$

$$\begin{aligned} J &= \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) (W_\epsilon \nabla u + r_\epsilon) \cdot \nabla_x D(x, y) dx \\ &= |\omega_\epsilon| \int_{\Omega} (\gamma_1 - \gamma_0) \frac{\chi_{\omega_\epsilon}(x)}{|\omega_\epsilon|} \frac{\partial w_\epsilon^i}{\partial x_j} \frac{\partial u}{\partial x_j} \frac{\partial D(x, y)}{\partial x_i} dx + \int_{\omega_\epsilon} (\gamma_1 - \gamma_0) r_\epsilon \cdot \nabla_x D(x, y) dx. \end{aligned}$$

As in [9, pp. 169], we have the following: Let $\Omega_1 \subset\subset \Omega$ denote a compact set that strictly contains the inhomogeneties ω_ϵ . Given $y \in \partial\Omega$, we can find a vector-valued function $\phi_y \in C^0(\overline{\Omega}_1)$ such that $\phi_y(x) = \nabla_x D(x, y)$ for all $x \in \Omega_1$. Also since u is smooth in the interior of Ω and dM_{ij}^θ is supported in a compact subset of Ω , we have

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} (\gamma_1 - \gamma_0) \frac{\chi_{\omega_\epsilon}(x)}{|\omega_\epsilon|} \frac{\partial w_\epsilon^i}{\partial x_j} \frac{\partial u}{\partial x_j} \frac{\partial \phi_y(x)}{\partial x_i} dx = \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij}^\theta \frac{\partial u}{\partial x_j} \frac{\partial D(x, y)}{\partial x_i} d\mu^\theta.$$

Since $r_\epsilon \rightarrow 0$ strongly in $(L_{\text{loc}}^1(\Omega))^N$, we have

$$\int_{\omega_\epsilon} (\gamma_1 - \gamma_0) r_\epsilon \cdot \nabla_x D(x, y) dx \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Hence

$$\begin{aligned} \gamma_0 \left(\frac{\partial u_\epsilon}{\partial \nu} - \frac{\partial u}{\partial \nu} \right)(y) &= |\omega_\epsilon| \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij}^\theta \frac{\partial u}{\partial x_j} \frac{\partial D(x, y)}{\partial x_i} d\mu^\theta \\ &\quad + \int_{\Omega} (\gamma_0 I - \gamma^*(x)) \nabla u(x) \cdot \nabla_x D(x, y) dx + o(1) \end{aligned}$$

where the $o(1)$ goes to zero uniformly in y as ϵ goes to zero. This completes the proof of Theorem 1.1. \blacksquare

2.1 Some properties of polarization tensor

PROPOSITION 2.2 (Relation between polarization and homogenization tensor). *Let $\delta > 0$. The polarization tensor and homogenization tensor are related as follows:*

$$\theta(x)(\gamma_1 - \gamma_0)M^\theta(x) = \gamma^*(x) - \gamma_0 I. \quad (12)$$

REMARK 2.4. *If $\delta = 0$, then above equality holds trivially because $\theta = 0$ almost everywhere and $\gamma^* = \gamma_0 I$. In some cases, above relation then degenerates.*

Proof. From (6), we have

$$\int_{\Omega} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \frac{\partial w_\epsilon^i}{\partial x_j}(x) \phi(x) dx \rightarrow \int_{\Omega} \phi(x) M_{ij}^\theta d\mu^\theta \quad \text{for all } \phi \in C^0(\overline{\Omega}).$$

Multiplying both sides by $(\gamma_1 - \gamma_0)$, we get

$$\int_{\Omega} (\gamma_1 - \gamma_0) \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \frac{\partial w_\epsilon^i}{\partial x_j}(x) \phi(x) dx \rightarrow \int_{\Omega} (\gamma_1 - \gamma_0) \phi(x) M_{ij}^\theta d\mu^\theta \quad \text{for all } \phi \in C^0(\overline{\Omega}).$$

Since $\gamma_\epsilon(x) = \chi_{\omega_\epsilon}(x)\gamma_1 + (1 - \chi_{\omega_\epsilon}(x))\gamma_0$, we have,

$$\int_{\Omega} (\gamma_1 - \gamma_0)\phi(x)M_{ij}^\theta d\mu^\theta = \lim_{\epsilon \rightarrow 0} \frac{1}{|\omega_\epsilon|} \int_{\Omega} (\gamma_\epsilon(x) - \gamma_0) \frac{\partial w_\epsilon^i}{\partial x_j} \phi(x) dx.$$

From (10),

$$\int_{\Omega} (\gamma_1 - \gamma_0)\phi(x)M_{ij}^\theta d\mu^\theta = \frac{1}{\int_{\Omega} \theta(x) dx} \int_{\Omega} (\gamma_{ij}^*(x) - \gamma_0 \delta_{ij}) \phi(x) dx.$$

Now since $d\mu^\theta = \frac{\theta(x)}{\int_{\Omega} \theta(x) dx} dx$ almost everywhere in Ω we obtain (12). This completes the proof. \blacksquare

REMARK 2.5 (Localization principle for the polarization tensor of the non zero volume fraction). *Let $\gamma_\epsilon \in M(\gamma_1, \gamma_0; \Omega)$ and $\tilde{\gamma}_\epsilon \in M(\gamma_1, \gamma_0; \Omega)$ be two sequences, which H -converge to $\gamma^*(x)$ and $\tilde{\gamma}^*(x)$, respectively. Let U be an open subset compactly embedded in Ω and if $\gamma_\epsilon(x) = \tilde{\gamma}_\epsilon(x)$ in U . It is known that $\gamma^*(x) = \tilde{\gamma}^*(x)$ in U with the same volume fraction θ in U . Thus it follows from (12) $M^\theta(x) = \widetilde{M}^\theta(x)$ in U .*

REMARK 2.6. *We don't have a similar property for polarization tensor corresponding to near zero volume fraction. Let $\gamma_\epsilon \in M(\gamma_1, \gamma_0; \Omega)$ and $\tilde{\gamma}_\epsilon \in M(\gamma_1, \gamma_0; \Omega)$ be two sequences with zero-volume fractions i.e. $|\omega_\epsilon|$ and $|\tilde{\omega}_\epsilon|$ goes to zero as ϵ tends to zero. Let U be an open subset compactly embedded in Ω and $\gamma_\epsilon(x) = \tilde{\gamma}_\epsilon(x)$ in U . Then if in addition $\frac{|\omega_\epsilon|}{|\tilde{\omega}_\epsilon|} \rightarrow 1$ as $\epsilon \rightarrow 0$, it follows from (8) and (9) that $M^0(x) = \widetilde{M}^0(x)$ and $\mu^0(x) = \tilde{\mu}^0(x)$ in U .*

REMARK 2.7. *Using Proposition 2.2 and (5), we can rewrite the asymptotic formula in Theorem 1.1 as follows:*

$$(\gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} - \gamma^* \frac{\partial u}{\partial \nu})(y) = (|\omega_\epsilon| - \delta) \int_{\Omega} (\gamma_1 - \gamma_0) M_{ij}^\theta(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial D}{\partial x_j}(x, y) d\mu^\theta(x) + o(1).$$

COROLLARY 2.1. *The polarization tensor M^θ is symmetric.*

Proof. This follows from (12) and from the fact that γ^* is symmetric. \blacksquare

2.2 Bounds on M^θ

We now derive bounds for the polarization tensor based on relation we obtained between the this tensor and the homogenization tensor (Proposition 2.2).

PROPOSITION 2.3. *We have the following bounds for the polarization tensor M^θ for μ^θ almost everywhere x*

$$\min \left\{ 1, \frac{\gamma_0}{\theta\gamma_0 + (1 - \theta)\gamma_1} \right\} |\xi|^2 \leq M_{ij}^\theta(x) \xi_i \xi_j \leq \max \left\{ 1, \frac{\gamma_0}{\theta\gamma_0 + (1 - \theta)\gamma_1} \right\} |\xi|^2, \quad \xi \in \mathbb{R}^N.$$

Proof. The proof is a straightforward application of Proposition 2.2 the arithmetic mean ($\gamma_a I$), harmonic mean ($\gamma_h I$) bound for γ^* [1, §2.1.2]:

$$\underbrace{\left(\frac{\theta(x)}{\gamma_1} + \frac{1-\theta(x)}{\gamma_0} \right)^{-1}}_{\gamma_h} I \leq \gamma^*(x) \leq \underbrace{(\theta(x)\gamma_1 + (1-\theta(x))\gamma_0)}_{\gamma_a} I.$$

■

The above result generalizes to the case of non-zero volume fraction of the conductivities, the bounds obtained at the conclusion of [9, Lemma 3].

PROPOSITION 2.4. *We have the following pointwise trace bounds for the polarization tensor M^θ for μ^θ almost everywhere x*

$$\text{Upper bound: } \text{trace} (I - \theta(x)M^\theta(x))^{-1} \leq \frac{N}{1-\theta(x)} + \frac{\theta(x)}{1-\theta(x)} \left(\frac{\gamma_0}{\gamma_1} - 1 \right), \quad (13)$$

$$\text{Lower bound: } \text{trace} (\theta(x)M^\theta(x))^{-1} \leq \frac{N}{\theta(x)} - \frac{1-\theta(x)}{\theta(x)} \left(1 - \frac{\gamma_1}{\gamma_0} \right). \quad (14)$$

Proof. The proof follows by slight modifications of the proof of [2, Prop. 3.1], [16] and the relation between the polarization tensor and the homogenization tensor (cf. Proposition 2.2). Recall that we have let $\gamma_\epsilon(x) = \chi_\epsilon(x)\gamma_1 + (1-\chi_\epsilon(x))\gamma_0$. Following the proof as in [2, Prop. 3.1], we have the following pointwise lower bound for $\gamma^*(x)$

$$(\gamma^*(x) - \gamma_1 I)^{-1} \leq \frac{I}{\gamma_a(x) - \gamma_1} + \frac{\theta(x)(1-\theta(x))(\gamma_1 - \gamma_0)^2}{\gamma_1(\gamma_a(x) - \gamma_1)^2} \widetilde{M}(x), \quad (15)$$

where \widetilde{M} [2, Eq. 12] is given by

$$\widetilde{M}(x) = \frac{1}{\theta(x)(1-\theta(x))} \int_{\mathbb{S}^{N-1}} \xi \otimes \xi \, d\nu(\xi).$$

Here $\theta(x)(1-\theta(x))dx \otimes d\nu(\xi)$ is the H -measure ([16, §28]) of the sequence $(\chi_{\phi_\epsilon} - \theta)(x)$. Note that \widetilde{M} has unit trace.

Now from Proposition 2.2, we have

$$\gamma^*(x) - \gamma_0 I = (\gamma_1 - \gamma_0)\theta(x)M^\theta(x).$$

Now substituting this into (15), we have

$$(I - \theta(x)M^\theta(x))^{-1} \leq \frac{\gamma_0 - \gamma_1}{\gamma_a(x) - \gamma_1} I + \frac{\theta(x)(1-\theta(x))(\gamma_0 - \gamma_1)^3}{\gamma_1(\gamma_a(x) - \gamma_1)^2} \widetilde{M}(x)$$

Since $\gamma_a(x) - \gamma_1 = (1-\theta(x))(\gamma_0 - \gamma_1)$ and then by taking trace, we get the inequality (13).

Similarly, by using upper bound for γ^* [2, Prop. 3.1], we deduce (14) which gives the lower bound for M^θ . ■

REMARK 2.8. *Proposition 2.2 combined with [1, Theorem 2.2.13] gives us that the pointwise trace bounds for the polarization tensor M^θ obtained in the previous proposition are optimal.*

3 Proof of Theorem 1.2 and Theorem 1.3

Proof of Theorem 1.2. Step 1 : Given $d\mu^0$, M^0 with $M^0 \in L^2(\Omega, d\mu^0)$, there exists a sequence of microstructures $\omega_\epsilon \subset K \subset \Omega$ (for some compact set K , as $\text{dist}(\omega_\epsilon, \partial\Omega) \geq d_0 > 0$) such that as $\epsilon \rightarrow 0$, $|\omega_\epsilon| \rightarrow 0$ and

$$\begin{aligned} (i) \quad & \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \xrightarrow{*} d\mu^0 \text{ in } (C^0(\overline{\Omega}))^*. \\ (ii) \quad & \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \frac{\partial w_\epsilon^i}{\partial x_j} dx \xrightarrow{*} dM_{ij}^0 = M_{ij}^0 d\mu^0 \text{ in } (C^0(\overline{\Omega}))^*. \end{aligned} \tag{1}$$

where w_ϵ^i are correctors defined in (10).

We consider any point $x_0 \in \text{Support of } (d\mu^0)$ and an open cube $Q_{x_0,h} = x_0 + (-\frac{h}{2}, \frac{h}{2})^N$ centered at the point x_0 which is included in K for sufficiently small $h > 0$. Note we can write $Q_{x_0,h} = x_0 + hY$, where the cell $Y = (-\frac{1}{2}, \frac{1}{2})^N$ with $|Y| = 1$. As $x_0 \in \text{Support of } (d\mu^0)$,

$$\mu^0(Q_{x_0,h}) > 0, \forall h > 0.$$

Making the change of variables

$$x \in Q_{x_0,h} \mapsto y \in Y \text{ as } x = x_0 + hy$$

we localize $d\mu^0$ and dM_{ij}^0 over $Q_{x_0,h}$.

Now following the idea that locally any inhomogeneous medium can be approximated by the periodic medium (cf. [1, Theorem 1.3.23]), let us consider the periodic homogenization with inhomogeneities $(\omega_\epsilon \cap Q_{x_0,h})$ contained in $Q_{x_0,h}$ and extending it periodically in whole \mathbb{R}^N with the small period $\eta > 0$. The corresponding coefficient is denoted by $\gamma_\epsilon(x_0 + h\frac{x}{\eta})$. Let $\gamma_{x_0,\epsilon,h}^*$ denote the homogenized tensor thus obtained. The volume fraction of inhomogeneities is evidently

$$\theta_{x_0,\epsilon,h} = \int_Y \chi_{\omega_\epsilon \cap Q_{x_0,h}}(x_0 + hy) dy = \frac{|\omega_\epsilon \cap Q_{x_0,h}|}{|Q_{x_0,h}|}. \tag{2}$$

The volume fraction $\theta_{x_0,\epsilon,h} > 0$ for ϵ being small enough, which follows from (6) below. We recall the integral representation of $\gamma_{x_0,\epsilon,h}^*$

$$(\gamma_{x_0,\epsilon,h}^*)_{ij} = \int_Y \gamma_\epsilon(x_0 + hy) \nabla \tilde{w}_{x_0,\epsilon,h}^i(y) \cdot e_j dy \tag{3}$$

where, $(\tilde{w}_{x_0,\epsilon,h}^i(y))_{1 \leq i \leq N}$ is the family of unique solutions in $H^1(Y)/\mathbb{R}$ of the cell problems

$$\nabla \cdot (\gamma_\epsilon(x_0 + hy) \nabla \tilde{w}_{x_0,\epsilon,h}^i(y)) = 0 \quad \text{in } Y, \quad y \mapsto (\tilde{w}_{x_0,\epsilon,h}^i(y) - y_i) \quad \text{is } Y \text{ periodic.} \tag{4}$$

In the sequel, we consider the polarization tensor denoted as $M^{\theta_{x_0,\epsilon,h}}(x_0)$ which corresponds to the above periodic microstructure. As observed in Remark 1.2, $M^{\theta_{x_0,\epsilon,h}}(x_0)$ is a constant

matrix. Using the relation (12) between the homogenized tensor and the polarization tensor with non-zero volume fraction $\theta_{x_0, \epsilon, h}$, the following integral representation is easily obtained from (3) :

$$\begin{aligned} M_{ij}^{\theta_{x_0, \epsilon, h}}(x_0) &= \int_Y \frac{1}{\theta_{x_0, \epsilon, h}} \chi_{\omega_\epsilon \cap Q_{x_0, h}}(x_0 + hy) \nabla_y \tilde{w}_{x_0, \epsilon, h}^i(y) \cdot e_j \, dy \\ &= \frac{1}{|\omega_\epsilon \cap Q_{x_0, h}|} \int_{Q_{x_0, h}} \chi_{\omega_\epsilon \cap Q_{x_0, h}}(x) h \nabla_x \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \cdot e_j \, dx. \end{aligned} \quad (5)$$

Our next task is to replace $\tilde{w}_{x_0, \epsilon, h}^i$ by w_ϵ^i in (5) as well as to analyze its limiting behavior as $\epsilon \rightarrow 0$ and $h \rightarrow 0$ in that order. In order to do that, we invoke our next step as follows.

Step 2 : We begin with defining limiting quantities representing volume fraction of inhomogeneities in $Q_{x_0, h}$ ($h > 0$) :

$$\limsup_{\epsilon \rightarrow 0} \frac{|\omega_\epsilon \cap Q_{x_0, h}|}{|\omega_\epsilon|} = V_h^+(x_0) \quad \text{and} \quad \liminf_{\epsilon \rightarrow 0} \frac{|\omega_\epsilon \cap Q_{x_0, h}|}{|\omega_\epsilon|} = V_h^-(x_0)$$

Clearly $V_h^+(x_0)$ and $V_h^-(x_0)$ are monotonically increasing function with respect to h , i.e.

$$V_h^+(x_0) \quad (\text{respectively, } V_h^-(x_0)) \geq V_{\tilde{h}}^+(x_0) \quad (\text{respectively, } V_{\tilde{h}}^-(x_0)), \quad \text{whenever } h \geq \tilde{h}.$$

Claim : For $h > 0$ fixed,

$$\mu^0(\overline{Q}_{x_0, h}) = V_h^+(x_0) = V_h^-(x_0). \quad (6)$$

REMARK 3.1. It follows that $\lim_{\epsilon \rightarrow 0} \frac{|\omega_\epsilon \cap Q_{x_0, h}|}{|\omega_\epsilon|}$ exists and equal to $\mu^0(\overline{Q}_{x_0, h}) > 0$.

Proof of the above claim : We will first show

$$V_h^+(x_0) \leq \mu^0(\overline{Q}_{x_0, h}).$$

Let us consider an open cube $Q_{x_0, h-\delta}$ for $\delta > 0$ small enough. Note $Q_{x_0, h-\delta}$ is relatively compact in $Q_{x_0, h}$. We consider a sequence of test functions $\phi_h^\delta \in C^0(\overline{\Omega})$ for h fixed such that

$$\text{Support of } \phi_h^\delta \subset Q_{x_0, h}, \quad \phi_h^\delta \equiv 1 \quad \text{in } Q_{x_0, h-\delta}, \quad \text{and} \quad 0 \leq \phi_h^\delta(x) \leq 1, \quad x \in Q_{x_0, h} \setminus \overline{Q}_{x_0, h-\delta}. \quad (7)$$

As we see for $h > 0$ fixed, as $\delta \downarrow 0$,

$$\phi_h^\delta(x) \rightarrow 1 \quad \text{everywhere } x \text{ in } Q_{x_0, h}.$$

Thus using the Lebesgue dominated convergence theorem and the fact μ^0 is a bounded measure

$$\mu^0(Q_{x_0, h}) = \int_{Q_{x_0, h}} d\mu^0 = \lim_{\delta \downarrow 0} \int_{Q_{x_0, h}} \phi_h^\delta d\mu^0.$$

So,

$$\mu^0(Q_{x_0, h}) = \lim_{\delta \downarrow 0} \int_{Q_{x_0, h}} \phi_h^\delta d\mu^0 = \lim_{\delta \downarrow 0} \int_{\Omega} \phi_h^\delta d\mu^0 = \lim_{\delta \downarrow 0} \left(\lim_{\epsilon \rightarrow 0} \int_{\Omega} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon} \phi_h^\delta dx \right),$$

$$\begin{aligned}
&\geq \limsup_{\delta \downarrow 0} \left(\limsup_{\epsilon \rightarrow 0} \int_{Q_{x_0, h-\delta}} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon} \phi_h^\delta dx \right), \\
&= \limsup_{\delta \downarrow 0} \left(\limsup_{\epsilon \rightarrow 0} \int_{Q_{x_0, h-\delta}} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon} dx \right), \\
&= \limsup_{\delta \downarrow 0} \left(\limsup_{\epsilon \rightarrow 0} \frac{|\omega_\epsilon \cap Q_{x_0, h-\delta}|}{|\omega_\epsilon|} \right), \\
&= \limsup_{\delta \downarrow 0} V_{h-\delta}^+(x_0). \tag{8}
\end{aligned}$$

Now as μ^0 is a Radon measure, for any given $\eta > 0$ there exist a $\delta_\eta > 0$ such that

$$\mu^0(\overline{Q}_{x_0, h}) + \eta \geq \mu^0(Q_{x_0, h+\delta_\eta}) \quad (\text{outer regularity}).$$

And using the above result (8) with $(h + \delta_\eta)$ in the place of h , we get

$$\mu^0(\overline{Q}_{x_0, h}) + \eta \geq \limsup_{\delta \downarrow 0} V_{h+\delta_\eta-\delta}^+(x_0).$$

Now as $0 < \delta < \delta_\eta$, by using the monotonicity we get $V_{h+\delta_\eta-\delta}^+(x_0) \geq V_h^+(x_0)$ for h being fixed. Consequently,

$$\mu^0(\overline{Q}_{x_0, h}) + \eta \geq V_h^+(x_0).$$

As $\eta > 0$ is chosen arbitrarily therefore

$$\mu^0(\overline{Q}_{x_0, h}) \geq V_h^+(x_0).$$

Similarly, we will show

$$\mu^0(\overline{Q}_{x_0, h}) \leq V_h^-(x_0).$$

Since the arguments are slightly different, we go through them. Here we will consider an open cube $Q_{x_0, h+\delta}$ for $\delta > 0$ small enough. Note that $Q_{x_0, h}$ is relatively compact in $Q_{x_0, h+\delta}$ for every $\delta > 0$. We consider a sequence of test functions $\psi_h^\delta \in C^0(\overline{\Omega})$ for h fixed, such that

$$\text{Support of } \psi_h^\delta \subset Q_{x_0, h+\delta}, \quad \psi_h^\delta \equiv 1 \text{ in } \overline{Q}_{x_0, h}, \quad \text{and } 0 \leq \psi_h^\delta(x) \leq 1, \quad x \in Q_{x_0, h+\delta} \setminus \overline{Q}_{x_0, h}. \tag{9}$$

So,

$$\left(\int_{\Omega} \psi_h^\delta d\mu^0 - \int_{\overline{Q}_{x_0, h}} d\mu^0 \right) = \int_{Q_{x_0, h+\delta} \setminus \overline{Q}_{x_0, h}} \psi_h^\delta d\mu^0 \leq \mu^0(Q_{x_0, h+\delta} - \overline{Q}_{x_0, h}) \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Thus

$$\mu^0(\overline{Q}_{x_0, h}) = \int_{\overline{Q}_{x_0, h}} d\mu^0 = \lim_{\delta \downarrow 0} \int_{\Omega} \psi_h^\delta d\mu^0 = \lim_{\delta \downarrow 0} \left(\lim_{\epsilon \rightarrow 0} \int_{\Omega} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon} \psi_h^\delta dx \right)$$

$$\begin{aligned}
&\leq \liminf_{\delta \downarrow 0} \left(\liminf_{\epsilon \rightarrow 0} \int_{Q_{x_0, h+\delta}} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon} dx \right) \\
&= \liminf_{\delta \downarrow 0} \left(\liminf_{\epsilon \rightarrow 0} \frac{|\omega_\epsilon \cap Q_{x_0, h+\delta}|}{|\omega_\epsilon|} \right) \\
&= \liminf_{\delta \downarrow 0} V_{h+\delta}^-(x_0). \tag{10}
\end{aligned}$$

Now as μ^0 is a Radon measure, for any given $\eta > 0$ there exists a $\delta_\eta > 0$ such that

$$\mu^0(\overline{Q}_{x_0, h}) - \eta \leq \mu^0(\overline{Q}_{x_0, h-\delta_\eta}) \quad (\text{inner regularity}).$$

And using the above result (10) with $(h - \delta_\eta)$ in the place of h , we get

$$\mu^0(\overline{Q}_{x_0, h}) - \eta \leq \liminf_{\delta \downarrow 0} V_{h-\delta_\eta+\delta}^-(x_0).$$

Now as $0 < \delta < \delta_\eta$, by using the monotonicity we get $V_{h-\delta_\eta+\delta}^-(x_0) \leq V_h^-(x_0)$ for h being fixed. It follows

$$\mu^0(\overline{Q}_{x_0, h}) - \eta \leq V_h^-(x_0).$$

As $\eta > 0$ is chosen arbitrarily therefore

$$\mu^0(\overline{Q}_{x_0, h}) \leq V_h^-(x_0).$$

Hence we have established our claim (6).

Step 3 : As the next step, in a very similar way as we just did in Step 2, we will show, for $h > 0$ fixed, that

$$\int_{\overline{Q}_{x_0, h}} dM_{ij}^0(x) = V_h^+(x_0) \cdot \limsup_{\epsilon \rightarrow 0} M_{ij}^{\theta_{x_0, \epsilon, h}}(x_0) = V_h^-(x_0) \cdot \liminf_{\epsilon \rightarrow 0} M_{ij}^{\theta_{x_0, \epsilon, h}}(x_0). \tag{11}$$

Let us first show that,

$$\int_{\overline{Q}_{x_0, h}} dM_{ij}^0(x) \geq V_h^+(x_0) \cdot \limsup_{\epsilon \rightarrow 0} M_{ij}^{\theta_{x_0, \epsilon, h}}(x_0)$$

Using the Lebesgue dominated convergence theorem together with the fact dM_{ij}^0 (cf. (1)) is a bounded measure we write first

$$\begin{aligned}
\int_{Q_{x_0, h}} dM_{ij}^0(x) &= \lim_{\delta \downarrow 0} \int_{Q_{x_0, h}} \phi_h^\delta dM_{ij}^0 = \lim_{\delta \downarrow 0} \int_{\Omega} \phi_h^\delta dM_{ij}^0 \\
&= \lim_{\delta \downarrow 0} \left(\lim_{\epsilon \rightarrow 0} \int_{\Omega} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \nabla_x w_\epsilon^i(x) \cdot e_j \phi_h^\delta(x) dx \right) \tag{12}
\end{aligned}$$

where, ϕ_h^δ is defined as in (7).

Our next task is to replace ω_ϵ by $\tilde{\omega}_{x_0, \epsilon, h}$ in the above representation.

Claim :

$$\begin{aligned} & \frac{1}{|\omega_\epsilon \cap Q_{x_0, h}|} \int_{Q_{x_0, h}} \chi_{\omega_\epsilon \cap Q_{x_0, h}}(x) \nabla_x w_\epsilon^i(x) \cdot e_j \phi_h^\delta(x) dx \\ &= \frac{1}{|\omega_\epsilon \cap Q_{x_0, h}|} \int_{Q_{x_0, h}} \chi_{\omega_\epsilon \cap Q_{x_0, h}}(x) h \nabla_x \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \cdot e_j \phi_h^\delta(x) dx + \mathcal{E}_{h, \phi_h^\delta}(\epsilon) \end{aligned}$$

where, $\mathcal{E}_{h, \phi_h^\delta}(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ for every fixed h and ϕ_h^δ . (13)

As we see, from (9) and (4) it follows that

$$\begin{aligned} & \gamma_1 \|\phi_h^\delta \left(\nabla_x w_\epsilon^i(x) - h \nabla_x \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \right)\|_{L^2(Q_{x_0, h})}^2 \\ & \leq \int_{Q_{x_0, h}} (\phi_h^\delta)^2 \gamma_\epsilon(x) \left(\nabla_x w_\epsilon^i(x) - h \nabla_x \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \right) \cdot \left(\nabla_x w_\epsilon^i(x) - h \nabla_x \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \right) dx \\ & = - \int_{Q_{x_0, h}} \left(\nabla_x \cdot \gamma_\epsilon(x) \nabla_x w_\epsilon^i(x) - \nabla_x \cdot \gamma_\epsilon(x) h \nabla_x \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \right) (\phi_h^\delta)^2 \left(w_\epsilon^i(x) - h \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \right) dx \\ & \quad - \int_{Q_{x_0, h}} \left(w_\epsilon^i(x) - h \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \right) \gamma_\epsilon(x) \left(\nabla_x w_\epsilon^i(x) - \nabla_x \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \right) \cdot \nabla_x (\phi_h^\delta)^2 dx \\ & = - \int_{Q_{x_0, h}} \gamma_\epsilon(x) \left(w_\epsilon^i(x) - h \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \right) 2 \nabla_x \phi_h^\delta \cdot \phi_h^\delta \left(\nabla_x w_\epsilon^i(x) - h \nabla_x \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \right) dx. \end{aligned}$$

Thus,

$$\begin{aligned} C \|\phi_h^\delta \left(\nabla_x w_\epsilon^i(x) - h \nabla_x \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \right)\|_{L^2(Q_{x_0, h})} & \leq \|w_\epsilon^i(x) - x_i\|_{L^2(Q_{x_0, h})} + \|h \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) - x_i\|_{L^2(Q_{x_0, h})}. \end{aligned}$$

Now, as it is shown in [9, Lemma 1.], we invoke

$$\begin{aligned} \|w_\epsilon^i(x) - x_i\|_{L^2(\Omega)} & \leq o(|\omega_\epsilon|^{\frac{1}{2}}) = o(|\omega_\epsilon \cap Q_{x_0, h}|^{\frac{1}{2}}) \quad (\text{by using the Remark 3.1}) \\ \text{and } \|h \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) - x_i\|_{L^2(Q_{x_0, h})} & \leq o(|\omega_\epsilon \cap Q_{x_0, h}|^{\frac{1}{2}}). \end{aligned} \tag{14}$$

Hence we get

$$|\mathcal{E}_{h, \phi_h^\delta}(\epsilon)| \leq \frac{1}{|\omega_\epsilon \cap Q_{x_0, h}|} \|\chi_{\omega_\epsilon \cap Q_{x_0, h}}(x) \phi_h^\delta \left(\nabla_x w_\epsilon^i(x) - h \nabla_x \tilde{w}_{x_0, \epsilon, h}^i\left(\frac{x - x_0}{h}\right) \right)\|_{L^1(Q_{x_0, h})}$$

$$\begin{aligned}
&\leq \frac{1}{|\omega_\epsilon \cap Q_{x_0,h}|^{\frac{1}{2}}} \|\phi_h^\delta \left(\nabla_x w_\epsilon^i(x) - h \nabla_x \tilde{w}_{x_0,\epsilon,h}^i\left(\frac{x-x_0}{h}\right) \right)\|_{L^2(Q_{x_0,h})} \\
&= o(1) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0 \quad (\text{by using (14) for being } h, \phi_h^\delta \text{ fixed}).
\end{aligned}$$

Thus from (12) and (13) we get

$$\begin{aligned}
\int_{Q_{x_0,h}} dM_{ij}^0(x) &= \lim_{\delta \downarrow 0} \left(\lim_{\epsilon \rightarrow 0} \int_{\Omega} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \nabla_x w_\epsilon^i(x) \cdot e_j \phi_h^\delta(x) dx \right) \\
&= \lim_{\delta \downarrow 0} \left(\lim_{\epsilon \rightarrow 0} \frac{|\omega_\epsilon \cap Q_{x_0,h}|}{|\omega_\epsilon|} \frac{1}{|\omega_\epsilon \cap Q_{x_0,h}|} \int_{Q_{x_0,h}} \chi_{\omega_\epsilon}(x) h \nabla_x \tilde{w}_\epsilon^i\left(\frac{x-x_0}{h}\right) \cdot e_j \phi_h^\delta(x) dx \right). \quad (15)
\end{aligned}$$

We show below the contribution coming from the annular region $Q_{x_0,h} \setminus Q_{x_0,h-\delta}$ is negligible. We may also replace $|\omega_\epsilon \cap Q_{x_0,h}|$ by $|\omega_\epsilon \cap Q_{x_0,h-\delta}|$ as shown below.

$$\begin{aligned}
&\frac{1}{|\omega_\epsilon \cap Q_{x_0,h}|} \int_{Q_{x_0,h} \setminus Q_{x_0,h-\delta}} \chi_{\omega_\epsilon}(x) h \nabla_x \tilde{w}_\epsilon^i\left(\frac{x-x_0}{h}\right) \cdot e_j \phi_h^\delta(x) dx \\
&\frac{1}{|\omega_\epsilon \cap Q_{x_0,h}|} \int_{Q_{x_0,h} \setminus Q_{x_0,h-\delta}} \chi_{\omega_\epsilon}(x) \left(h \nabla_x \tilde{w}_\epsilon^i\left(\frac{x-x_0}{h}\right) - e_i + e_i \right) \cdot e_j \phi_h^\delta(x) dx \\
&\leq C \frac{1}{|\omega_\epsilon \cap Q_{x_0,h}|} \left(|\omega_\epsilon \cap Q_{x_0,h} \setminus Q_{x_0,h-\delta}|^{\frac{1}{2}} \right) \cdot \left(|\omega_\epsilon \cap Q_{x_0,h}|^{\frac{1}{2}} + |\omega_\epsilon \cap Q_{x_0,h} \setminus Q_{x_0,h-\delta}|^{\frac{1}{2}} \right) \\
&\quad \quad \quad (\text{using (4), cf. Remark (2.1)}) \\
&= C \left(\left(1 - \frac{|\omega_\epsilon \cap Q_{x_0,h-\delta}|}{|\omega_\epsilon \cap Q_{x_0,h}|}\right)^{\frac{1}{2}} + \left(1 - \frac{|\omega_\epsilon \cap Q_{x_0,h-\delta}|}{|\omega_\epsilon \cap Q_{x_0,h}|}\right) \right). \quad (16)
\end{aligned}$$

Now as we see that,

$$\begin{aligned}
\lim_{\delta \downarrow 0} \left(\lim_{\epsilon \rightarrow 0} \frac{|\omega_\epsilon \cap Q_{x_0,h-\delta}|}{|\omega_\epsilon \cap Q_{x_0,h}|} \right) &= \lim_{\delta \downarrow 0} \left(\lim_{\epsilon \rightarrow 0} \frac{|\omega_\epsilon \cap Q_{x_0,h}|/|\omega_\epsilon|}{|\omega_\epsilon \cap Q_{x_0,h-\delta}|/|\omega_\epsilon|} \right) = \lim_{\delta \downarrow 0} \left(\frac{\mu^0(\overline{Q}_{x_0,h})}{\mu^0(\overline{Q}_{x_0,h-\delta})} \right) = 1. \\
&\quad \quad \quad (\because \mu^0 \text{ is a Radon measure})
\end{aligned} \quad (17)$$

So, (16) goes to zero as $\epsilon \rightarrow 0$ and $\delta \downarrow 0$ in that order, for every fixed $h > 0$.

Thus from (15),(16) and (17) it follows that,

$$\int_{Q_{x_0,h}} dM_{ij}^0(x) = V_h^+(x_0) \lim_{\delta \downarrow 0} \left(\limsup_{\epsilon \rightarrow 0} M_{ij}^{\theta_{x_0,\epsilon,h-\delta}} \right). \quad (18)$$

-the last line follows from (5) with $(h - \delta)$ in the place of h .

Now using the fact that $\gamma_{x_0,\epsilon,h}^*$ is scale invariant, it follows that

$$\gamma_{x_0,\epsilon,h}^* = \gamma_{x_0,\epsilon,h-\delta}^*.$$

So, using the relation (12) it follows that

$$\theta_{x_0, \epsilon, h} M^{\theta_{x_0, \epsilon, h}} = \theta_{x_0, \epsilon, h-\delta} M^{\theta_{x_0, \epsilon, h-\delta}}$$

Next using the expression (2) and as it is shown in (17), it simply follows that

$$\lim_{\delta \downarrow 0} \lim_{\epsilon \rightarrow 0} \frac{\theta_{x_0, \epsilon, h}}{\theta_{x_0, \epsilon, h-\delta}} = 1.$$

Thus (18) becomes

$$\int_{Q_{x_0, h}} dM_{ij}^0(x) = V_h^+(x_0) \left(\limsup_{\epsilon \rightarrow 0} M_{ij}^{\theta_{x_0, \epsilon, h}} \right).$$

Since dM_{ij}^0 is a Radon measure, so it follows that

$$\int_{\overline{Q}_{x_0, h}} dM_{ij}^0(x) \geq V_h^+(x_0) \left(\limsup_{\epsilon \rightarrow 0} M_{ij}^{\theta_{x_0, \epsilon, h}} \right).$$

Similarly, one shows (likewise as we did in Step 2 for $d\mu^0$)

$$\int_{\overline{Q}_{x_0, h}} dM_{ij}^0(x) \leq V_h^-(x_0) \left(\liminf_{\epsilon \rightarrow 0} M_{ij}^{\theta_{x_0, \epsilon, h}} \right).$$

We consider the sequence of test functions $\psi_h^\delta \in C^0(\overline{\Omega})$ defined in (9) and using the fact dM^0 is a Radon measure we get

$$\left| \int_{\Omega} \psi_h^\delta dM_{ij}^0 - \int_{\overline{Q}_{x_0, h}} dM_{ij}^0 \right| = \left| \int_{Q_{x_0, h+\delta} \setminus \overline{Q}_{x_0, h}} \psi_h^\delta dM_{ij}^0 \right| \leq \left| dM_{ij}^0(Q_{x_0, h+\delta} \setminus \overline{Q}_{x_0, h}) \right| \rightarrow 0, \text{ as } \delta \rightarrow 0.$$

Thus

$$\begin{aligned} \int_{\overline{Q}_{x_0, h}} dM_{ij}^0 &= \lim_{\delta \downarrow 0} \int_{\Omega} \psi_h^\delta dM_{ij}^0 = \lim_{\delta \downarrow 0} \left(\lim_{\epsilon \rightarrow 0} \int_{\Omega} \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(x) \nabla w_\epsilon^i(x) \cdot e_j \psi_h^\delta dx \right) \\ &= \lim_{\delta \downarrow 0} \left(\lim_{\epsilon \rightarrow 0} \frac{|\omega_\epsilon \cap Q_{x_0, h+\delta}|}{|\omega_\epsilon|} \frac{1}{|\omega_\epsilon \cap Q_{x_0, h+\delta}|} \int_{Q_{x_0, h+\delta}} \chi_{\omega_\epsilon}(x) \nabla \tilde{w}_{x_0, \epsilon, h+\delta}^i(x) \cdot e_j \psi_h^\delta dx \right). \end{aligned} \quad (19)$$

Again one shows the contribution coming from the annular region $Q_{x_0, h+\delta} \setminus Q_{x_0, h}$ is negligible. We may also replace $|\omega_\epsilon \cap Q_{x_0, h}|$ by $|\omega_\epsilon \cap Q_{x_0, h+\delta}|$, similar to as it is shown in (16) and (17). Therefore, from (19) it follows that

$$\int_{\overline{Q}_{x_0, h}} dM_{ij}^0(x) = V_h^-(x_0) \liminf_{\delta \downarrow 0} \left(\limsup_{\epsilon \rightarrow 0} M_{ij}^{\theta_{x_0, \epsilon, h-\delta}} \right). \quad (20)$$

-the last line follows from (5) with h in the place of $h + \delta$.

Thus (11) follows.

Step 4 : From (11), it follows, since $V_h^+(x_0) = V_h^-(x_0) = \mu^0(\overline{Q}_{x_0,h}) > 0$, $\lim_{\epsilon \rightarrow 0} M_{ij}^{\theta_{x_0,\epsilon,h}}$ exists and we have

$$\lim_{\epsilon \rightarrow 0} M_{ij}^{\theta_{x_0,\epsilon,h}} = \frac{1}{\mu^0(\overline{Q}_{x_0,h})} \int_{\overline{Q}_{x_0,h}} dM_{ij}^0(x)$$

In this final step, we will be passing to the limit as $h \rightarrow 0$ in the above relation to get the desired result. We use the Lebesgue differentiation theorem [6, Theorem 8.4.6.] to have

$$\lim_{h \rightarrow 0} \frac{1}{\mu^0(\overline{Q}_{x_0,h})} \int_{\overline{Q}_{x_0,h}} dM_{ij}^0(x) = \lim_{h \rightarrow 0} \frac{1}{\mu^0(\overline{Q}_{x_0,h})} \int_{\overline{Q}_{x_0,h}} M_{ij}^0(x) d\mu^0 = M_{ij}^0(x_0),$$

μ^0 almost everywhere x_0 .

Therefore, we finally get

$$\lim_{h \rightarrow 0} \lim_{\epsilon \rightarrow 0} M_{ij}^{\theta_{x_0,\epsilon,h}}(x_0) = M_{ij}^0(x_0), \quad \mu^0 \text{ almost everywhere } x_0.$$

In other words, there exist a sequence $\{\theta_{x_0}^n\}_{n \in \mathbb{N}} (> 0)$ depending upon the point x_0 , such that as $n \rightarrow \infty$, $\theta_{x_0}^n \rightarrow 0$ and the constant polarization tensors $M^{\theta_{x_0}^n} \rightarrow M^0(x_0)$, μ^0 almost everywhere x_0 . ■

REMARK 3.2. In the above proof, we have dealt with the convergence behavior of the polarization tensor $M^{\theta_{x_0,\epsilon,h}}$. Regarding the associated measure $\mu^{\theta_{x_0,\epsilon,h}}$, we have

$$\lim_{h \rightarrow 0} \left(\lim_{\epsilon \rightarrow 0} d\mu^{\theta_{x_0,\epsilon,h}} \right) = \delta_{x_0},$$

because, we have (from (5))

$$d\mu^{\theta_{x_0,\epsilon,h}} = \frac{\theta_{x_0,\epsilon,h}}{\int_{Q_{x_0,h}} \theta_{x_0,\epsilon,h} dx} dx = \frac{1}{|Q_{x_0,h}|} dx \quad (\text{independent of } \epsilon).$$

■

Theorem 1.2 immediately gives the pointwise bounds on the polarization tensor M^0 .

Proof of Theorem 1.3. Let's recall from Proposition 2.3 the pointwise bounds on the polarization tensor for $M^\theta(x)$ for μ^θ almost everywhere $x \in \Omega$ we have,

$$\min \left\{ 1, \frac{\gamma_0}{\theta\gamma_0 + (1-\theta)\gamma_1} \right\} |\xi|^2 \leq M_{ij}^\theta(x) \xi_i \xi_j \leq \max \left\{ 1, \frac{\gamma_0}{\theta\gamma_0 + (1-\theta)\gamma_1} \right\} |\xi|^2, \quad \xi \in \mathbb{R}^N. \quad (21)$$

Now take any point $x_0 \in \text{support of } d\mu^0$, we want to derive bounds on $M^0(x_0)$, μ^0 almost everywhere. By the previous Theorem 1.2, there exist a sequence $\theta_{x_0}^n (> 0)$ depending upon the point x_0 and a sequence of constant polarization tensors $M^{\theta_{x_0}^n}$ such that as $n \rightarrow \infty$, $\theta_{x_0}^n \rightarrow 0$ and $M^{\theta_{x_0}^n} \rightarrow M^0(x_0)$, μ^0 almost everywhere x_0 . Therefore, we have the following estimate,

$$\min \left\{ 1, \frac{\gamma_0}{\gamma_1} \right\} |\xi|^2 \leq M_{ij}^0(x_0) \xi_i \xi_j \leq \max \left\{ 1, \frac{\gamma_0}{\gamma_1} \right\} |\xi|^2, \quad \xi \in \mathbb{R}^N.$$

It immediately shows that $M^0(x_0)$ is a positive definite matrix. Next we recall the optimal bounds on $M^\theta(x)$ from the Proposition 2.4.

$$\text{trace } (\theta_{x_0}^n M^{\theta_{x_0}^n})^{-1} \leq \frac{N}{\theta_{x_0}^n} - \frac{1 - \theta_{x_0}^n}{\theta_{x_0}^n} \left(1 - \frac{\gamma_1}{\gamma_0}\right).$$

This corresponds to the lower curve $c_1^{\theta_{x_0}^n}$ shown in Figure 1. Passing to the limit as n tends to infinity, we simply obtain the lower bound for $M^0(x_0)$ as

$$\text{trace } (M^0(x_0))^{-1} \leq (N - 1) + \frac{\gamma_1}{\gamma_0}.$$

Similarly for upper bound we recall from Proposition 2.4

$$\text{trace } (I - \theta_{x_0}^n M^{\theta_{x_0}^n})^{-1} \leq \frac{N}{1 - \theta_{x_0}^n} + \frac{\theta_{x_0}^n}{1 - \theta_{x_0}^n} \left(\frac{\gamma_0}{\gamma_1} - 1\right)$$

This corresponds to the upper curve $c_2^{\theta_{x_0}^n}$ shown in Figure 1. Since $\theta_{x_0}^n \rightarrow 0$, it follows that

$$\text{trace } (I + \theta_{x_0}^n M^{\theta_{x_0}^n}) \leq N(1 + \theta_{x_0}^n) + \theta_{x_0}^n (1 - \theta_{x_0}^n) \left(\frac{\gamma_0}{\gamma_1} - 1\right).$$

Thus as n tends to infinity we get the upper bound of $M^0(x_0)$ as

$$\text{trace } (M^0(x_0)) \leq (N - 1) + \frac{\gamma_0}{\gamma_1}.$$

This ends the proof of Theorem 1.3. The curves defined by (14) are shown in Figure 1 (in red). ■

In general, for a given density function $\theta \in L^\infty(\Omega; [0, 1])$, we denote by \mathcal{G}_θ (the G -closure set) the set of all possible H-limits

$$\mathcal{G}_\theta = \{\gamma^* \in L^\infty(\Omega; M(\gamma_1, \gamma_0; \Omega)) \mid \text{there exists a characteristic function } \chi_{\omega_\epsilon}(x) \text{ satisfying (3) and } \gamma_\epsilon(x) \text{ defined by (1), satisfies (4)}\}.$$

Similarly, we define \mathcal{M}_θ is the set of all possible polarization tensors corresponding to the density function $\theta(x)$ as

$$\mathcal{M}_\theta = \{M^\theta \text{ satisfying (21)} \mid M^\theta \text{ is defined by (6) and (7)}\}.$$

And we denote by \mathcal{M}_0 the set of all polarization tensors with the near zero volume fraction.

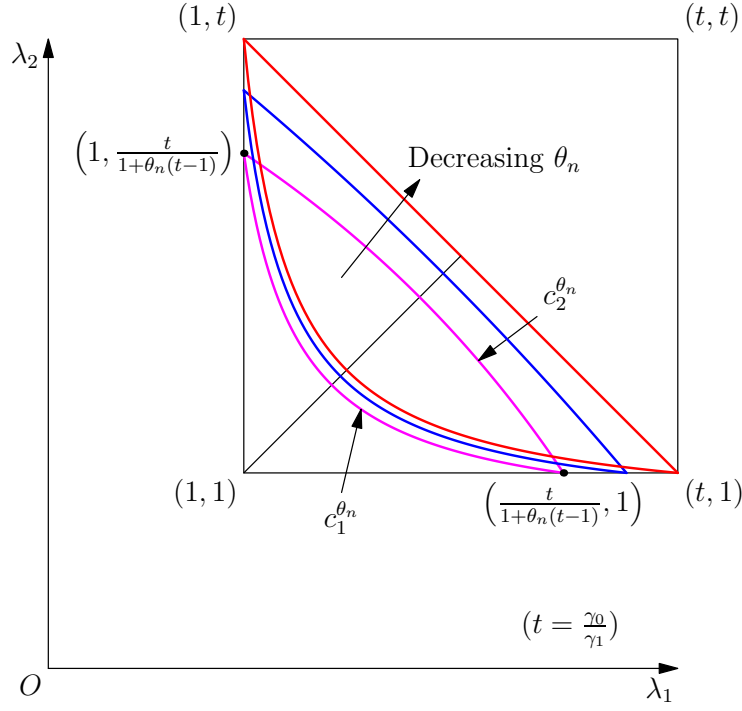


Figure 1: $N = 2$: The \mathcal{M}_0 region enclosed by pair of red curves can be seen as an approximation of the regions \mathcal{M}_θ enclosed by the pair of pink and pair of blue curves respectively as θ goes to zero.

REMARK 3.3. *If x_0 is such that the polarization tensor $M^0(x_0)$ satisfies the strict inequality $\text{trace}(M^0(x_0)) < (N - 1) + \frac{\gamma_0}{\gamma_1}$, then there exists a $\tilde{\theta} \in (0, 1]$ such that $M^0(x_0) = M^\theta \in \mathcal{M}_\theta$, $\forall \theta \leq \tilde{\theta}$. The existence of such $\tilde{\theta}$ is clear from the above Figure 1.*

4 Optimality of the trace bounds for M^0

We have derived the bounds for the polarization tensor M^0 in Section 3 (cf. Theorem 1.3(a)). These bounds were obtained earlier in [12, 4]. Here we prove the converse namely Theorem 1.3 (b). More precisely, we show that given $\{\lambda_1(x), \dots, \lambda_N(x)\}$ satisfying the inequalities (14), there are sequential laminates whose associated polarization tensor $M^0(x)$ has eigen values $\{\lambda_1(x), \dots, \lambda_N(x)\}$.

We begin by computing the polarization tensor for rank- p sequential laminates based on the relationship (12). The homogenization tensors for such laminates are well-known [1, 16].

EXAMPLE 4.1 (Rank- p Sequential Laminates). *[1, pp. 102] Let $(e_i)_{1 \leq i \leq p}$ be a collection of unit vectors in \mathbb{R}^N and $(\theta_i)_{1 \leq i \leq p}$ the proportions at each stage of the lamination process.*

Then we have the following :

(a): For a rank- p sequential laminate with matrix γ_0 and core γ_1

$$\left(\prod_{j=1}^p \theta_j\right)(\gamma_p^* - \gamma_0 I)^{-1}(x) = (\gamma_1 - \gamma_0)^{-1}I + \sum_{i=1}^p \left((1 - \theta_i) \left(\prod_{j=1}^{i-1} \theta_j\right) \right) \frac{e_i \otimes e_i}{\gamma_0}.$$

(b): For rank- p sequential laminate with matrix γ_1 and core γ_0

$$\left(\prod_{j=1}^p (1 - \theta_j)\right)(\gamma_p^* - \gamma_1 I)^{-1}(x) = (\gamma_1 - \gamma_0)^{-1}I + \sum_{i=1}^p \left(\theta_i \left(\prod_{j=1}^{i-1} (1 - \theta_j)\right) \right) \frac{e_i \otimes e_i}{\gamma_1}.$$

The following lemma from is important for proving the optimality.

LEMMA 4.1. [1] Let $(e_i)_{1 \leq i \leq p}$ be a collection of unit vectors. Let $\theta \in (0, 1]$. Now for a fixed point x and for any collection of non-negative real numbers $(m_i)_{1 \leq i \leq p}$ satisfying $\sum_{i=1}^p m_i = 1$, there exists a rank- p sequential laminate $\gamma_p^*(x)$ with matrix $\gamma_0 I$ and core $\gamma_1 I$ in proportion $(1 - \theta)$ and θ respectively and with lamination directions $(e_i)_{1 \leq i \leq p}$ such that

$$\theta(\gamma_p^* - \gamma_0 I)^{-1}(x) = (\gamma_1 - \gamma_0)^{-1}I + (1 - \theta) \sum_{i=1}^p m_i \frac{e_i \otimes e_i}{\gamma_0 e_i \cdot e_i}. \quad (1)$$

An analogous result holds when the roles of γ_0 and γ_1 (in proportions $(1 - \theta)$ and θ respectively) in the lemma above are switched. The formula above is replaced by

$$(1 - \theta)(\gamma_p^* - \gamma_1 I)^{-1}(x) = (\gamma_0 - \gamma_1)^{-1}I + \theta \sum_{i=1}^p m_i \frac{e_i \otimes e_i}{\gamma_1 e_i \cdot e_i}. \quad (2)$$

■

Denoting the polarization tensor M_p^θ in the case of matrix $\gamma_0 I$ and core $\gamma_1 I$, by (12) and (1) we then have,

$$(M_p^\theta(x))^{-1} = I + (1 - \theta) \frac{(\gamma_1 - \gamma_0)}{\gamma_0} \sum_{i=1}^p m_i (e_i \otimes e_i). \quad (3)$$

For matrix $\gamma_1 I$ and core $\gamma_0 I$ from (12) and (2), we have

$$(I - \theta M_p^\theta(x))^{-1} = \frac{1}{(1 - \theta)} I + \frac{\theta}{1 - \theta} \frac{(\gamma_0 - \gamma_1)}{\gamma_1} \sum_{i=1}^p m_i (e_i \otimes e_i) \quad (4)$$

■

Proof of Theorem 1.3. Let $M(x_0)$ be a second order positive definite tensor with its eigenvalues $(\lambda_1(x_0), \dots, \lambda_N(x_0))$ lying on the curve defining the upper bound in the Figure 1 :

$$1 \leq \lambda_i(x_0) \leq \frac{\gamma_0}{\gamma_1}, \quad 1 \leq i \leq N \quad \text{and} \quad \sum_{i=1}^N \lambda_i(x_0) = (N-1) + \frac{\gamma_0}{\gamma_1}. \quad (5)$$

Then there exists a collection of non-negative real numbers $(m_i)_{1 \leq i \leq N}$ depending upon the point x_0 satisfying

$$\sum_{i=1}^N m_i = 1 \quad \text{and} \quad \lambda_i(x_0) = 1 + m_i \left(\frac{\gamma_0}{\gamma_1} - 1 \right). \quad (6)$$

After getting such $(m_i)_{1 \leq i \leq N}$'s through the equation (6), and choosing $\theta \in (0, 1]$ we consider the rank- N sequential laminated structure defined in the Lemma 4.1. We obtain a polarization tensor in M_N^θ with matrix γ_1 and core γ_0 such that

$$(I - \theta M_N^\theta)(x_0)^{-1} = \frac{1}{(1-\theta)} I + \frac{\theta}{1-\theta} \frac{(\gamma_0 - \gamma_1)}{\gamma_1} \sum_{i=1}^N m_i (e_i \otimes e_i).$$

The eigenvalues of $M_N^\theta(x_0)$ denoted by $(\lambda_i^\theta(x_0))_{1 \leq i \leq N}$ are given by

$$\frac{1}{1 - \theta \lambda_i^\theta(x_0)} = \frac{1}{(1-\theta)} + \frac{\theta}{1-\theta} \frac{(\gamma_0 - \gamma_1)}{\gamma_1} m_i. \quad (7)$$

Since m_i satisfies $0 \leq m_i \leq 1$, it follows

$$1 \leq \lambda_i^\theta(x_0) \leq \frac{\gamma_0}{\theta(\gamma_0 - \gamma_1) + \gamma_1}$$

and since $\sum_{i=1}^N m_i = 1$, we have also

$$\sum_{i=1}^N (1 - \theta \lambda_i^\theta(x_0))^{-1} = \frac{N}{(1-\theta)} + \frac{\theta}{1-\theta} \frac{(\gamma_0 - \gamma_1)}{\gamma_1}.$$

Now choosing a sequence $\theta_n \rightarrow 0$ (e.g: $\theta_n = \frac{1}{n}$), we will show as $n \rightarrow \infty$ $\lambda_i^{\theta_n}(x_0) \rightarrow \lambda_i(x_0)$. From the asymptotic expansion of the equation (7) in terms of $\theta \approx 0$ we have

$$(1 + \theta_n \lambda_i^{\theta_n}(x_0)) \approx (1 + \theta_n) + \theta_n (1 + \theta_n) \frac{(\gamma_0 - \gamma_1)}{\gamma_1} m_i.$$

So as $n \rightarrow \infty$,

$$\lambda_i^{\theta_n}(x_0) \rightarrow 1 + \frac{(\gamma_0 - \gamma_1)}{\gamma_1} m_i = \lambda_i(x_0).$$

This is equivalent to saying

$$M_N^{\theta_n}(x_0) \rightarrow M(x_0).$$

To finish the proof we need to show $M \in \mathcal{M}_0$ (see the end of Section 3 for the definition of \mathcal{M}_0) To this end, since we are working at the point x_0 , we can in fact assume that we are dealing with periodic rank- N sequential laminates. Now the result quickly follows [4, Theorem 8.1].

Similarly one shows that equality of lower bound can be achieved through the N sequential laminates with matrix γ_0 and core γ_1 .

To show that any interior point in the region defined by the bounds (14) corresponds to a polarization tensor of near zero volume fraction, we can follow the arguments found in [1, Page no.124-125].

Thus, any tensor $M(x)$ satisfying the pointwise bounds given by (14) is in \mathcal{M}_0 , which completes our discussion on optimality of the bounds. ■

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